ALGEBRAIC K-THEORY WITH APPLICATIONS TO MODEL THEORY

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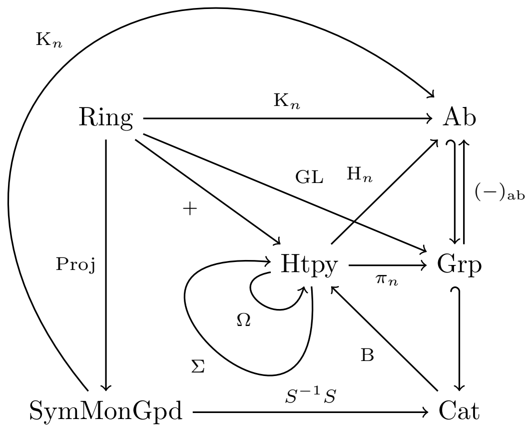

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1. Introduction & Motivation

Algebraic K-theory provides a sequence K_* of Ab-valued covariant functors on Ring or, more generally, on small exact categories or symmetric monoidal groupoids.

Grothendieck's study of algebraic K-theory was motivated by Serre's topological K-theory. We define the (topological) $K^0(X)$ to be the group completion of stable isomorphism classes of vector bundles on X. This is a contravariant functor Top $\to$ *Ab*.

Definition 1.1. *Let E be an $\mathbb{F}$-vector bundle on X. A global section of E is a continuous function $s : X \to E$ such that $p \circ s = Id_X$. The set of all global sections of E can be made into a $C_{\mathbb{F}}(X)$-module, $\Gamma(E)$ by pointwise addition and scalar multiplication.*

Theorem 1.2 (Swan's Theorem)**.** *Let X be a compact, Hausdorff topological space and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. $\Gamma(-)$ is an equivalence of categories $\mathrm{Vect}_{\mathbb{F}}(X) \to \mathrm{Proj}(C_{\mathbb{F}}(X))$.*

Corollary 1.3. *For a compact Hausdorff topological space X, $K^0(X) \cong K_0(C_{\mathbb{C}}(X))$ and $KO^0(X) \cong K_0(C_{\mathbb{R}}(X))$.*

2. Lower K-theory

Given a ring R, the lower K-theory classifies:

- K_0: finite rank projective modules over that ring under $\oplus$;
- K_1: automorphisms of those projective modules under composition.

Definition 2.1 (Finitely generated projective module)**.** *For our purposes, a **finitely generated projective R-module** over a ring R is an R-module P satisfying $P \oplus Q \cong R^n$ for some R-module Q and some n.*

Definition 2.2 (K_0 of a ring)**.** *The set $\mathrm{Proj}\text{-}R$ of isomorphism classes of finitely generated projective R-modules is a commutative monoid, with associative binary operation $\oplus$ (extended to classes) and identity element $[0]$. We define $\mathrm{K}_0(R)$ to be the abelian group of stable isomorphism classes of finitely generated projective R-modules, where $P_1, P_2 \in \mathrm{Proj}\text{-}R$ are **stably isomorphic** if $P_1 \oplus R^n \cong P_2 \oplus R^n$ for some n.*

Definition 2.3 (Grothendieck group)**.** *The **Grothendieck group** $\mathrm{G}(M)$ of a commutative monoid M is the universal group completion of M. It can be constructed as the quotient of $M \times M$ by the equivalence relation*

$$(x, y) \sim (u, v) \quad \Leftrightarrow \quad x + v + t = u + y + t \text{ for some } t \in S. \tag{1}$$

Remark 2.4. Note that $\mathrm{K}_0(R) = \mathrm{G}(\mathrm{Proj}\text{-}R)$.

Theorem 2.5 (K_0 as a functor)**.** *$\mathrm{K}_0(-)$ is a functor $\mathrm{Ring} \to \mathrm{Ab}$ and we have $\mathrm{K}_0(R \times S) \cong \mathrm{K}_0(R) \oplus \mathrm{K}_0(S)$ for all rings R and S.*

If R is a commutative ring then every R-module is naturally a bimodule, so $\mathrm{K}_0(R)$ then becomes a ring with multiplication $\otimes_R$. In this case K_0 is a functor $\mathrm{CRing} \to \mathrm{CRing}$ and we have $\mathrm{K}_0(R \times S) \cong \mathrm{K}_0(R) \times \mathrm{K}_0(S)$ as rings for all rings R and S.

Definition 2.6 (K_1 of a ring)**.** *First define $\mathrm{GL}(R)$ to be the union of all the groups $\mathrm{GL}_n(R)$. This set is a group, with multiplication of $A, B \in \mathrm{GL}(R)$ given by first embedding A and B into the top left corner of identity matrices of the same size. We then define $\mathrm{K}_1(R)$ to be the group $\mathrm{GL}(R)/\,\mathrm{E}(R)$, where $\mathrm{E}(R)$ is the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ of $\mathrm{GL}(R)$. That is, $\mathrm{K}_1(R)$ is the maximal abelian quotient $\mathrm{GL}(R)_{\mathrm{ab}}$ of $\mathrm{GL}(R)$.*

Theorem 2.7 (Morita invariance)**.** *If R and S are Morita equivalent rings (i.e., the module categories $\mathrm{Mod}\text{-}R$ and $\mathrm{Mod}\text{-}S$ are equivalent), then they have isomorphic K-theory. In particular, we have $\mathrm{K}_i(\mathrm{M}_n\, R) \cong \mathrm{K}_i(R)$ for $i = 0, 1$.*

Examples 2.8.
1. $\mathrm{G}(\mathbf{N}) \cong \mathbf{Z}$.
2. If R is a division ring, a principal ideal domain or a local ring then every finitely generated projective R-module is isomorphic to R^n for some unique n. This sets up a correspondence between $\mathrm{Proj}\text{-}R$ and $\mathbf{N}$, so $\mathrm{K}_0(R) \cong \mathbf{Z}$.

(3) If V is an infinite dimensional vector space over a field F then $\mathrm{K}_0(\mathrm{End}_F V) = 0$ because we have the relation $[\mathrm{End}_F] \oplus [\mathrm{End}_F] = [\mathrm{End}_F]$ in Proj-R. This example is known as the "Eilenberg swindle."
(4) If R is a Dedekind domain then $\mathrm{K}_0(R) \cong \mathbf{Z} \oplus \mathrm{C}(R)$, where $\mathrm{C}(R)$ is the class group of R. For example, we have $\mathrm{K}_0\big(\mathbf{Z}[-\sqrt{5}]\big) \cong \mathbf{Z} \oplus \mathbf{Z}/2$.
(5) If R is a local ring then there is a unique determinant map $\mathrm{GL}(R) \to R^{\times}_{\mathrm{ab}}$, which induces an isomorphism $\mathrm{K}_1(R) \cong R^{\times}_{\mathrm{ab}}$. In particular, if F is a field then we have $\mathrm{K}_1(F) \cong F^{\times}$.

Algebraic K-theory behaves quite a lot like a homology theory. There is a natural way of building the K-theory of the ring R given $K_*(R/I)$ and 'relative' K-groups $K_*(R, I)$ for any two-sided ideal $I \lhd R$.

$$K_1(R,I) \to K_1(R) \to K_1(R/I) \to K_0(R,I) \to K_0(R) \to K_0(R/I) \tag{2}$$

The class of regular rings is 'nicely behaved' with respect to algebraic homotopy.

Definition 2.9. *A ring R is said to be (left)* ***regular*** *if it is (left) noetherian and every finitely generated (left) R-module has a finite type projective resolution.*

Examples of such rings include every PID and every Dedekind domain, whereas group rings $\mathbb{Z}G$ of non-trivial finite groups G are not regular.

Theorem 2.10 (Grothendieck's Theorem)**.** *Let R be a left regular ring. Then there are natural isomorphisms* $\mathrm{K}_0(R) \cong \mathrm{K}_0(R[t]) \cong \mathrm{K}_0(R[t,t^{-1}])$ *and* $\mathrm{K}_1(R) \cong \mathrm{K}_1(R[t])$.

Theorem 2.11 (Bass-Heller-Swan)**.** *Let R be a left regular ring. Then there is a natural isomorphism* $\mathrm{K}_1(R[t,t^{-1}]) \cong \mathrm{K}_1(R) \oplus \mathrm{K}_0(R)$.

Motivated by Bass-Heller-Swan, we inductively define the negative K-groups as follows. For all $n \geq 0$ we let,

$$\mathrm{K}_{-n-1}(R) := \mathrm{coker}(\mathrm{K}_{-n}(R[t]) \oplus \mathrm{K}_{-n}(R[t^{-1}]) \to \mathrm{K}_{-n}(R[t,t^{-1}])).$$

Furthermore, we define $\mathrm{NK}_{-n}(R) := \mathit{coker}(\mathrm{K}_{-n}(R) \to \mathrm{K}_{-n}(R[t]))$.

Theorem 2.12 (Fundamental Theorem of Algebraic K-theory)**.** *For any ring R and any $n \in \mathbb{N}$ there is a natural splitting,*

$$\mathrm{K}_{-n}(R[t,t^{-1}]) \cong \mathrm{K}_{-n}(R) \oplus \mathrm{K}_{-n+1}(R) \oplus \mathrm{NK}_{-n}(R) \oplus \mathrm{NK}_{-n}(R)$$

3. Geometric Realization of a Category

Throughout this section let C be a small category[1].

Definition 3.1. *The nerve, NC, of C, is the simplicial set whose n-simplices are diagrams c in C of the form,*

$$c_0 \to c_1 \to \cdots \to c_n. \tag{3}$$

The i^{th} face of the simplex $\delta_i(c)$ is obtained by deleting c_i in the obvious manner, and the i^{th} degeneracy $\sigma_i(c)$ is obtained by replacing c_i with $c_i \xrightarrow{=} c_i$. The geometric realization of NC, denoted $|NC|$ is the CW-complex obtained by attaching an n-cell via its boundary to each non-degenerate n-simplex and making identifications according to the boundary maps. We define the geometric realization BC of C to be the geometric realization of its nerve.

Category theoretic properties of C match well with the homotopy theoretic properties of BC.

[1] we can also apply the definitions of this section to a category C with small skeleton C_0, in which case BC_0 is well defined upto homotopy

Proposition 3.2. *Any natural transformation $\eta : F_1 \to F_2$ between functors $F_1, F_2 : C \to D$ induces a homotopy $B\eta : BC \times [0,1] \to BD$ between the maps BF_1 and BF_2.*

Corollary 3.3. *Any adjoint pair of functors $F : C \to D$ and $G : D \to C$ induce a homotopy equivalence between BC and BD. In particular, any equivalence of categories C, D induces a homotopy between BC and BD, and for any category C with a terminal (or initial) object, BC is contractible.*

The following result is a tool to determine whether a functor $F : C \to D$ is a homotopy equivalence.

Theorem 3.4 (Quillen's Theorem A)**.** *Let $F : C \to D$ be a functor such that $d\backslash F$ is contractible for every $d \in D$. The $BF : BC \to BD$ is a homotopy equivalence.*

We summarize some definitions from topology needed to obtain a long exact sequence of homotopy groups.

Definition 3.5. *A continuous map $f : E \to B$ induces maps $\pi_* E \to \pi_* B$. The **homotopy fiber** $F(f)$ of f, relative to a basepoint $*_B$ of B, is the space of pairs (e, γ), where $e \in E$ and $\gamma : [0,1] \to B$ is a path starting at $*_B$ and ending at $f(e)$. A sequence of based spaces $F \to E \xrightarrow{f} B$ with $F \to B$ constant is called a **homotopy fibration sequence** if the map $F \to F(f)$ is a homotopy equivalence.*

Given a basepoint $*_E$ with $f(*_E) = *_B$, there is a long exact sequence of homotopy groups.

$$\begin{aligned} \ldots \to \pi_{n+1} B \to \pi_n F(f) \to \pi_n E \to \pi_n B \to \pi_{n-1} F(f) \to \ldots \\ \to \pi_1 B \to \pi_0 F(f) \to \pi_0 E \to \pi_0 B. \end{aligned} \tag{4}$$

The main result of this section is the following categorical analogue of the homotopy fiber sequence.

Theorem 3.6 (Quillen's Theorem B)**.** *Let $F : C \to D$ be a functor such that for every morphism $d \to d'$ in D the induced functor $d'\backslash F \to d\backslash F$ is a homotopy equivalence. Then for each $d \in D$ the geometric realization of the sequence,*

$$d\backslash F \xrightarrow{j} C \xrightarrow{F} D,$$

is a homotopy fibration sequence. Thus there is a long exact sequence of homotopy groups,

$$\cdots \to \pi_{i+1}(BD) \xrightarrow{\partial} \pi_i(Bd\backslash F) \xrightarrow{j} \pi_i(BC) \xrightarrow{F} \pi_i(BD) \xrightarrow{\partial} \ldots$$

4. Quillen's + Construction

Algebraic topology studies functors π_n, H_n : **Top** $\to$ **Ab**; the motivation behind lies in the fact that algebraic objects are sometimes simpler to handle than topological spaces. Quillen(1969) proposed a definition of higher K-groups of the ring R as the homotopy groups of a certain topological space $BGL(R)^+$. This is an 'almost functorial' construction (i.e., upto homotopy equivalence of topological spaces). The properties that such a space is expected to have are mentioned in the following definition.

Definition 4.1. *The notation $BGL(R)^+$ will denote any CW-complex X which has a distinguished map $BGL(R) \to X$ such that*

1. *$\pi_1 X \cong K_1(R)$, and the natural map from $GL(R) = \pi_1 BGL(R)$ to $\pi_1 X$ is onto with kernel $E(R)$,*
2. *$H_*(BGL(R); M) \xrightarrow{\cong} H_*(X; M)$ for every $K_1(R)$-module M.*

*In this case, we say that X is a **model** for $BGL(R)^+$. Write $K(R)$ for the product $K_0(R) \times BGL(R)^+$ and define $K_n(R) := \pi_n K(R)$.*

The topological space $K(R)$ is rigged so that the earlier definitions of $K_i(R)$ for $i = 0, 1, 2$ agree with the new definition.

Definition 4.2. *A topological space X is said to be **acyclic** if it has the homology of a point and a map $f : X \to Y$ is said to be acyclic if its homotopy fiber is acyclic.*

The **Volodin space** $X(R)$ of a ring R is an acyclic subspace of $BGL(R)$ given by $X(R) := \bigcup_{n=1}^{\infty} \bigcup_{\sigma \in \Sigma_n} BT_n(R)^{\sigma}$, where T_n is the subgroup of $GL_n(R)$ consisting of upper triangular matrices with 1 on the diagonal and Σ_n is the permutation group on n elements represented by matrices. The image of the map $\pi_1 X(R) \to \pi_1 BGL(R)$ is the group $E(R)$. The space $BGL(R)/X(R)$ is a model for $BGL(R)^+$.

We need more results from topology of acyclic spaces to obtain an algebraic version of the $+$-construction. The proof of the following result uses Serre spectral sequence.

Lemma 4.3. *If E is an acyclic space, then E is connected and $\pi_1 E$ is perfect group with $H_2(\pi_1 E; \mathbb{Z}) = 0$.*

Note from the last few terms of the sequence(4) that if $X \to Y$ is acyclic, then $\pi_1 X \to \pi_1 Y$ is onto with a perfect normal subgroup of $\pi_1 X$ as its kernel. This motivates the following definition.

Definition 4.4. *Let P be a perfect normal subgroup of $\pi_1 X$, where X is a based connected CW-complex. An acyclic map $f : X \to Y$ is called a **$+$-construction** on X relative to P if P is the kernel of $\pi_1 X \to \pi_1 Y$.*

Every group has a largest perfect normal subgroup called as its perfect radical. The following theorem was proved by Quillen in 1969.

Theorem 4.5. *(Quillen) Let P be a perfect normal subgroup of $\pi_1(X)$. Then*

1. *there is a $+$-construction $f : X \to Y$ relative to P,*
2. *if $f : X \to Y$ is a $+$-construction relative to P and $g : X \to Z$ is a map such that P vanishes in $\pi_1(Z)$, then there is a map $h : Y \to Z$, unique upto homotopy, such that $g = hf$,*
3. *furthermore if g is another $+$-construction relative to P, then the map h is a homotopy equivalence.*

The following lemma helps to connect the theorem with the definition of $BGL(R)^+$, whose proof uses Serre spectral sequence.

Lemma 4.6. *Let X and Y be connected CW-complexes. A map $f : X \to Y$ is acyclic if and only if $H_*(X, M) \cong H_*(Y, M)$ for every $\pi_1(Y)$-module M.*

Several other constructions of models of $BGL(R)^+$ are known and some of them are functorial.

The higher K-groups are graded components of a ring, where the multiplication is defined as follows. If A and B are rings, then there is a tensor product homomorphism $GL_n(A) \times GL_m(B) \to GL_{n+m}(A \otimes B)$. These maps induce a map $\gamma : BGL(A)^+ \wedge BGL(B)^+ \to BGL(A \otimes B)^+$, well-defined upto weak homotopy equivalence. Thus we get a product map $K_p(A) \otimes K_q(B) \to K_{p+q}(A \otimes B)$.

Theorem 4.7. *(Loday) The product map is natural in A and B, bilinear and associative. If A is commutative, then the induced product $K_p(A) \otimes K_q(A) \to K_{p+q}(A \otimes A) \to K_{p+q}(A)$ is graded-commutative.*

Theorem 4.8 (Fundamental theorem)**.** *Let R be a ring. Then there is a canonically split exact sequence*

$$(5)\qquad \begin{array}{c} 0 \longrightarrow \mathrm{K}_n(R) \longrightarrow \mathrm{K}_n(R[t]) \oplus \mathrm{K}_n\big(R\big[t^{-1}\big]\big) \\ \downarrow \\ \mathrm{K}_n\big(R[t,t^{-1}]\big) \xrightarrow{\partial} \mathrm{K}_{n-1}(R) \longrightarrow 0, \end{array}$$

where the splitting of ∂ is given by multiplication by t. Moreover, if R is a regular ring then we have

$$(6)\qquad \mathrm{K}_n\big(R\big[t,t^{-1}\big]\big) \cong \mathrm{K}_n(R) \oplus \mathrm{K}_{n-1}(R).$$

Some known higher K-groups

Higher K-groups are very hard to compute and the current state of knowledge of groups, even for fields, in incomplete.

Theorem 4.9 (Quillen)**.** *$K_{2k}(\mathbb{F}_q) = 0$ and $K_{2k-1}(\mathbb{F}_q) = C_{q^k-1}$ for all $k > 0$.*

Theorem 4.10. *Let F be an algebraically closed field of characteristic 0. Then $K_{2k}(F)$ is uniquely divisible and $K_{2k-1}(F)$ is a direct sum of a uniquely divisible group and a torsion group $\mathbb{Q}/\mathbb{Z}$ for all $k > 0$.*

5. Definable Combinatorics and Grothendieck Rings

Let L denote any language and M denote any first order L-structure. The term definable will always mean definable with parameters from M.

Definitions 5.1. *For every $n \geq 1$, we define $\mathrm{Def}(M^n)$ to be the collection of all definable subsets of M^n. We define $\overline{\mathrm{Def}}(M) := \bigcup_{n\geq 1} \mathrm{Def}(M^n)$.*

Definition 5.2. *We say that two definable sets $A, B \in \overline{\mathrm{Def}}(M)$ are **definably isomorphic** if there exists a definable bijection between them, i.e., a bijection $f : A \to B$ such that the graph $Gr(f) \in \overline{\mathrm{Def}}(M)$. This is an equivalence relation on $\overline{\mathrm{Def}}(M)$ and the equivalence class of a set A is denoted by $[A]$. We use $\widetilde{\mathrm{Def}}(M)$ to denote the set of all equivalence classes with respect to this relation. We use $[-] : \overline{\mathrm{Def}}(M) \to \widetilde{\mathrm{Def}}(M)$ to denote the surjective map defined by $A \mapsto [A]$.*

We can regard $\widetilde{\mathrm{Def}}(M)$ as an L_{ring}-structure. In fact, it is a semiring with respect to the operations defined as follows:

- $0 := [\emptyset]$;
- $1 := [\{*\}]$ for any singleton subset $\{*\}$ of M;
- $[A] + [B] := [A' \sqcup B']$ for $A' \in [A], B' \in [B]$ such that $A' \cap B' = \emptyset$;
- $[A] \cdot [B] := [A \times B]$.

This following ring captures some aspects of the definable combinatorics of the structure M.

Definition 5.3. *We define the **model-theoretic Grothendieck ring of the first order structure** M, denoted by $K_0(M)$, to be the abelian group $G(\widetilde{\mathrm{Def}}(M))$ equipped with multiplication induced by the cartesian product of the definable sets, where G denotes the Grothendieck group construction of 2.3.*

Definition 5.4. *We say that an infinite structure M satisfies the **onto pigeonhole principle** if for each $A \in \overline{\mathrm{Def}}(M)$ and each definable injection $f : A \rightarrowtail A$, we have $f(A) \neq A \setminus \{a\}$ for any $a \in A$.*

Proposition 5.5. *Given any infinite structure M, $K_0(M) \neq \{0\}$ if and only if M satisfies the onto pigeonhole principle.*

Examples 5.6. Here is a list of some known Grothendieck Rings.

- If M is a finite structure, then $K_0(M) \cong \mathbb{Z}$.
- (Krajíček-Scanlon) If $\mathbb{R}$ is a real closed field, then $K_0(\mathbb{R}) \cong \mathbb{Z}$.
- (R. Cluckers and D. Haskell) $K_0(\mathbb{F}_q((t))) = 0$.
- (R. Cluckers and D. Haskell) $K_0(\mathbb{Q}_p) = 0$.
- (Denef-Loeser) $\mathbb{Z}[u, v]$ is a quotient of $K_0(\mathbb{C})$.
- (Krajíček-Scanlon) $\mathbb{Z}[X_i : i \in \mathfrak{c}] \subseteq K_0(\mathbb{C})$.
- (Perera) If M is a vector space over a skew field R, then $K_0(M) \cong \mathbb{Z}[X]$.

Proposition 5.7 (Properties of Grothendieck rings). (1) *If M, N are L-structures and $M \preceq N$ then $K_0(M) \leq K_0(N)$.*

(2) *If $M \equiv N$, then $\widetilde{\mathrm{Def}}(M) \equiv_{\exists_1} \widetilde{\mathrm{Def}}(N)$ in L_{ring}. As the Grothendieck ring $K_0(M)$ is existentially interpretable in $\widetilde{\mathrm{Def}}(M)$, we have $K_0(M) \equiv_{\exists_1} K_0(N)$.*

(3) *If $M_{\mathcal{R}} \equiv N_{\mathcal{R}}$, then $K_0(M_{\mathcal{R}}) \cong K_0(N_{\mathcal{R}})$. Hence the Grothendieck ring of a module is an invariant of its theory.*

(4) *If $M_{\mathcal{R}}$ is nonzero, then there is a split embedding $\mathbb{Z} \to K_0(M_{\mathcal{R}})$ of rings. In particular, the Grothendieck ring is nontrivial.*

(5) *If M is a right $\mathcal{R}_1 \times \mathcal{R}_2$-module with natural decomposition $M = M_1 \oplus M_2$ with M_i a right $\mathcal{R}_i$-module, then $K_0(M) \cong K_0(M_1) \otimes_{\mathbb{Z}} K_0(M_2)$.*

Definition 5.8. *For a commutative monoid $(\mathcal{U}, \star, 1)$, the* ***integral monoid ring*** *$(\mathbb{Z}[\mathcal{U}], 0, 1, +, \cdot)$ is defined as follows.*

- $\mathbb{Z}[\mathcal{U}] := \{\phi : \mathcal{U} \to \mathbb{Z} :$ *the set Supp*$(\phi) = \{a : \phi(a) \neq 0\}$ *is finite*$\}$
- $(\phi + \psi)(a) := \phi(a) + \psi(a)$ *for* $a \in \mathcal{U}$
- $(\phi \cdot \psi)(a) := \Sigma_{b \star c = a} \phi(b)\psi(c)$ *for* $a \in \mathcal{U}$

Theorem 5.9. [2, Theorem 5.2.3] *Let $\mathcal{R}$ denote a unital ring and $M_{\mathcal{R}}$ denote an $\mathcal{R}$-module. Suppose $\overline{\mathcal{X}}$ is the multiplicative monoid of pp-isomorphism class of pp-sets. Then $K_0(M_{\mathcal{R}})$ is the quotient of the integral monoid ring $\mathbb{Z}[\overline{\mathcal{X}}]$ by the invariants ideal $\mathcal{J}$ that encodes finite indices of pp-pairs.*

6. K-theory of Symmetric Monoidal Groupoids

Definition 6.1. *A* ***groupoid*** *is a (skeletally) small category S in which every morphism is an isomorphism. A triple $(S, *, e)$ is a* ***symmetric monoidal groupoid*** *if the groupoid S is equipped with a bifunctor $* : S \times S \to S$, and a distinguished object e such that, for all objects $s, t, u \in S$, there are natural coherent isomorphisms*

$$e * s \cong s \cong s * e, \quad s * (t * u) \cong (s * t) * u, \quad s * t \cong t * s.$$

Definition 6.2. *Suppose S is symmetric monoidal, and suppose S^{iso} denotes the set of isomorphism classes of objects of S. Then S^{iso} is a commutative monoid with respect to the product induced by $*$, with identity element e. The Grothendieck group of this commutative monoid is denoted $K_0^*(S)$, or just $K_0(S)$ if the product is clear from the context.*

Examples 6.3. Let G be a finite group, and G-$\mathrm{Set}_{\mathrm{fin}}$ be the category of finite G-sets. This is symmetric monoidal with respect to $\coprod$, and $K_0(G\text{-}\mathrm{Set}_{\mathrm{fin}})$ is the *Burnside Ring* of G, often denoted $A(G)$. The category $\mathrm{Rep}_{\mathbb{C}}(G)$ of complex representations of G is symmetric monoidal with respect to $\oplus$. Applying K_0 to $\mathrm{Rep}_{\mathbb{C}}(G)$ produces the *representation ring* of G, $R(G)$.

An **H-space** is a topological space X with a continuous binary operation $\mu \colon X \times X \to X$ such that there is a point $e \in X$ for which the functions $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity on X, through homotopies preserving the point e. The geometric realization BS is an H-space with a homotopy-commutative and homotopy-associative product. The commutative monoid $\pi_0(S)$ is just the set of isomorphism classes of objects in S. In fact, S is equivalent to $\coprod Aut_S(s)$ and hence $B(S)$ is homotopy equivalent to the disjoint union of classifying spaces $B\,Aut_S(s)$, $s \in S^{iso}$.

Examples 6.4. The space $B(iso\,\mathrm{FinSet})$ is homotopy equivalent to $\coprod_{n\geq 0} B\Sigma_n$, where Σ_n is the permutation group on the set of n-elements.

The space $B(iso\,\text{Proj-}R)$ is homotopy equivalent to $\coprod B\,Aut(P)$, where P runs over isomorphism classes.

If $\mathbf{F}(R)$ is the category $\coprod_n GL_n(R)$, whose objects are the based free R-modules, then the space $B\mathbf{F}(R)$ is equivalent to $\coprod_n BGL_n(R)$.

We say that **translations are faithful** in S if every translation $Aut(s) \to Aut(s*t)$ in S is an injection. If translations are faithful in S, then the following construction of the category $S^{-1}S$ gives a "group completion" $B(S^{-1}S)$ of BS. The motivation comes from the group completion of an commutative monoid.

Definition 6.5. *If S is any symmetric monoidal category, then we define a new category $S^{-1}S$ as follows. The objects are pairs of objects of S and a morphism is an equivalence class of composites $(m_1, m_2) \xrightarrow{s*} (s*m_1, s*m_2) \xrightarrow{(f,g)} (n_1, n_2)$, where equivalence is induced by isomorphisms $\alpha : s \to t$ so that $(f, g) \circ (\alpha * m_i) = (f', g')$. This assignment is functorial for strict monoidal functors.*

The category $S^{-1}S$ is a symmetric monoidal category and the natural map $BS \to B(S^{-1}S)$ is an H-space map. It induces a map of monoids $\pi_0(S) \to \pi_0(S^{-1}S)$, where the target is an abelian group owing to the existence of a morphism $\eta : (e, e) \to (m, n) * (n, m)$.

Definition 6.6. *If S is any symmetric monoidal category, then we define the K-**theory space** $K^*(S)$ of S as the geometric realization of $S^{-1}S$. The K-groups of S are defined as $K_n^*(S) := \pi_n K^*(S)$.*

This assignment is functorial and the definition of $K_0^*(S)$ agrees with the earlier definition of $K_0(S)$ i.e., the group completion of $\pi_0(S^{iso})$.

A **pairing** of symmetric monoidal categories is a functor $\otimes : S_1 \times S_2 \to S$ such that $\otimes$ preserves identities and satisfies a (coherent) bi-distributivity law over $*$. Examples include $\times$ on FinSets and the tensor product of based free modules.

Peter May proved a theorem which states that a pairing determines a natural pairing of infinite loop spaces $K(-)$, as well as a pairing of Ω-spectra $\mathbf{K}(-)$, which in turn induces bilinear products on K-groups. These products agree with Loday's theorem for FinSets and $\mathbf{F}(R)$.

Quillen's $+$-construction relates with $S^{-1}S$ construction as stated in the following theorem.

Theorem 6.7. *When $S = \coprod GL_n(R)$, $K(S) = B(S^{-1}S)$ is the group completion of $BS = \coprod BGL_n(R)$, and $B(S^{-1}S) \simeq \mathbb{Z} \times BGL(R)^+$.*

Theorem 6.8. *(Barratt-Priddy-Quillen-Segal) The following three infinite loop spaces are the same:*

(a) *the group completion $K(\mathrm{FinSets})$ of $B\mathrm{FinSets}$,*
(b) $\mathbb{Z} \times B\Sigma^\infty$,
(c) $\Omega^\infty S^\infty = \lim_{n\to\infty} \Omega^n S^n$.

Hence the groups $K_n(\mathrm{FinSets})$ are the stable homotopy groups of spheres, π_n^s.

The free R-module functor $\mathrm{FinSets} \to \mathbf{F}(R)$ induces maps $\pi_n^s \to K_n(R)$.

7. K-theory of Model-Theoretic Structures

Let M denote an L-structure. Let $\mathrm{Def}(M^n)$ denote the set of definable subsets (with parameters from M) of M^n for each $n \geq 1$. Further let $\mathcal{S}$ (or $\mathcal{S}(M)$ if two or more structures are present) denote the category whose objects are $\bigcup_{n \in \mathbb{Z}_+} \mathrm{Def}(M^n)$ and whose maps are definable bijections between definable subsets. Then $(\mathcal{S}, \sqcup, \emptyset, \times)$ is a symmetric monoidal category where $\sqcup$ denotes disjoint union and the cartesian product $\times$ of definable sets is a pairing. Furthermore, translations are faithful in $\mathcal{S}$.

We define $K_n(M) := K_n(\mathcal{S})$ for each $n \geq 0$, where $K_n(\mathcal{S}) = \pi_n B\mathcal{S}^{-1}\mathcal{S}$ is the definition of K-groups of a symmetric monoidal groupoid given by Quillen.

Theorem 7.1 (Bass)**.** *If $\mathcal{S}$ is a symmetric monoidal groupoid whose translations are faithful, then*

$$K_1(\mathcal{S}) = \varinjlim_{S \in \mathcal{S}} H_1(\mathrm{Aut}_{\mathcal{S}}(S); \mathbb{Z}). \tag{7}$$

Note that if $\mathcal{S} = \mathcal{S}(M)$ for some structure M, then this theorem gives $K_1(M) = \mathrm{colim}_{n \to \infty} H_1(\mathrm{Aut}_{\mathcal{S}}(M^n); \mathbb{Z})$.

The following remark will be useful for computations.

Remark 7.2. Suppose that $(\mathcal{S}, \star, E)$ is a symmetric monoidal groupoid whose translations are faithful. Further suppose that $\mathcal{S}$ has a countable sequence of objects $S_1, S_2, \ldots$ such that $S_{n+1} = S_n \star A_n$ for some $A_n \in \mathcal{S}$, and satisfying the cofinality condition that for every $S \in \mathcal{S}$ there is an S' and an n such that $S \star S' \cong S_n$. In this case we can form the group $\mathrm{Aut}(\mathcal{S}) = \mathrm{colim}_{n \to \infty} \mathrm{Aut}_{\mathcal{S}}(S_n)$.

If M is an elementary substructure of N and $\phi(\overline{x}, \overline{a})$ is any formula with parameters from M, then $\phi(N, \overline{a})$ defines a subset of $N^{|\overline{x}|}$. This defines a (strict) monoidal functor $\mathcal{S}(M) \to \mathcal{S}(N)$. Hence K-theory is functorial on elementary embeddings.

Example 7.3. Let M be a finite structure. Then every subset of M^n is definable with parameters. Thus the symmetric monoidal groupoid $(\mathcal{S}, \sqcup, \emptyset)$ is equivalent to the groupoid $(\mathrm{FinSets}^{\mathrm{iso}}, \sqcup, \emptyset)$ of finite sets and bijections. Hence by Barratt-Priddy-Quillen-Segal theorem, $K_n(\mathcal{S})$ is the n^{th} stable homotopy group of spheres. In particular, $K_1(M) = \mathbb{Z}_2$.

8. K-theory of vector spaces

Let F denote an infinite field and V denote an infinite F-vector space. We study V as a structure in the language of F-vector spaces. The theory of V eliminates quantifiers in this language and hence every object of $\mathcal{S}(V_F)$ is a finite boolean combination of the basic definable subsets - the *pp*-definable subsets - namely $V, V^2, \ldots$ and their cosets in higher dimensional spaces.

To compute $K_1(\mathcal{S}(V_F)) := \varinjlim_{n \in \mathbb{N}} \mathrm{Aut}_{\mathcal{S}}(V^n)^{ab}$, we express $\mathrm{Aut}_{\mathcal{S}}(V^n)$ as an iterated semidirect product of certain wreath products.

Definition 8.1. *Let $(G, \cdot, e)$ be a group, $N \lhd G$ and $H \leq G$. If $G = NH$ and $N \cap H = \{e\}$, then we say that G is a* ***semidirect product*** *of N and H and write $G = H \ltimes N$.*

Let K and L be groups and let $B := \bigoplus_{l \in L} K_l$. The ***(restricted) wreath product*** *$K \wr L$ is defined to be the group $L \ltimes B$ and B is said to be the* ***base*** *of the wreath product.*

For example, the group $K \wr \Sigma_n$ is the set of n-tuples of elements of K with the permutation group acting on the indices in the tuples.

Proposition 8.2. *Suppose H acts on N. Then $(H \ltimes N)^{ab} = H^{ab} \times (N^{ab})_H$, where $(N^{ab})_H$ is the quotient of N^{ab} by the subgroup generated by elements of the form $n^h - n$ where n^h denotes the action of $h \in H$ on $n \in N^{ab}$ induced by the action of H on N.*

Proposition 8.3. $\mathrm{Aut}_{\mathcal{S}}(V^n)^{ab} = \Upsilon^n \ltimes (\Upsilon^n_{n-1} \ltimes (\Upsilon^n_{n-2} \ltimes (\cdots(\Upsilon^n_1 \ltimes \Upsilon^n_0)\cdots)))$, *where, for each $n > m \geq 0$, $\Upsilon^n_m := (\mathrm{GL}_n(F) \ltimes V^n) \wr \Sigma^n_m$ and Σ^n_m is the finitary permutation group on the set of cosets of an m-dimensional subspace of V^n.*

Theorem 8.4. *$K_0(V_F) = \mathbb{Z}[x] = \mathbb{Z}[\mathbb{N}]$, where $x = [V_F] \in K_0(V_F)$ and $K_1(V_F) = \mathbb{Z}_2 \oplus \bigoplus_{n\geq 1}(F^\times \oplus \mathbb{Z}_2)$.*

Conjecture 8.5. Suppose M is an infinite $\mathcal{R}$-module satisfying $M \equiv M \oplus M$. Then $K_1(M_{\mathcal{R}}) = \bigoplus_{\mathfrak{A}\in\mathcal{X}}(\Upsilon(\mathfrak{A})^{ab} \oplus \mathbb{Z}_2)$, where $\Upsilon(\mathfrak{A})$ is the group of *pp*-definable automorphisms of any *pp*-set in $\mathfrak{A}$.

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