Definable henselian valuations

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Let \((K, \mathcal{O})\) be an henselian valued field. In this poster we discuss the definability of \(\mathcal{O}\) as a subset of \(K\) using formulas from the language of rings. We are mainly interested in the special cases when the formula is existential or universal, and also when one formula gives a uniform definition for a family of valued fields.

We write ‘\(\exists \)-A-definable’ as shorthand for ‘definable with an existential formula using parameters from \(A\)’. Similarly for ‘\(\forall \)-A-definable’, etc.

**History and motivation**

The subject of defining valuations has a long history. Papers by Köngsman and Jahnke-Köningmann give very general conditions for an henselian valuation ring to be \(\exists \)-definable.

Returning to existential definitions, a recent paper of Cluckers-Derakhshan-Leenknegt-Macintyre contained several results, including a uniform existential definition (in an expansion of the ring language by a ‘Macintyre predicate’) for all henselian valued fields with finite or pseudofinite residue field.

Our focus is on existential and universal definitions in the language of rings, and our work continues a line of research which began with the following two theorems.

**Fact (Anscombe-Köngsman, [1])**

Let \(q\) be any prime power. Then \(\mathcal{O}_q[t]\) is \(\exists \)-definable in \(\mathcal{O}_q((t))\).

**Fact (Fehm, [3])**

Let \(F\) be any finite or PAC field which does not contain an algebraically closed field. Then \(F[[t]]\) is \(\exists \)-definable in \(F((t))\).

**Value groups don’t matter**

Our first theorem simplifies the question by removing the value group from consideration.

**Theorem**

Let \(F\) be a field of characteristic zero. Consider the family of valued fields \((K, \mathcal{O})\) with residue field elementarily equivalent to \(F\). Then either these valuations are uniformly \(\exists \)-definable or none of them is \(\exists \)-definable.

**Proof idea.**

Use the Ax-Kochen/Ershov principle: the theory of an equal characteristic henselian valued field is determined by its theories of its value group and residue field.

So the question of whether or not \((K, \mathcal{O})\) has an existential or universal definition depends only on the residue field. For simplicity, results will be stated about power series fields \(F((t))\), but will apply uniformly to all henselian valued fields with residue field \(F\).

**Existential definitions: negative results**

The first proposition is the main tool we use to prove negative results: it gives a condition on \(F\) which is sufficient for \(F[[t]]\) to not be \(\exists \)-definable in \(F((t))\).

**Proposition**

If \(F\) is of characteristic zero and admits a nontrivial valuation with residue field isomorphic to a subfield of \(F\), then there is no \(\exists \)-definition for \(F[[t]]\) in \(F((t))\).

**Proof idea.**

Self-embeddings preserve \(\exists \)-definable sets. We construct a self-embedding of \(F((t))\) so that the image of \(F\) is not contained in \(F[[t]]\).

The following easy lemma turns out to be very useful.

**Lemma**

Suppose that \(F \preceq K\) and that \(E[[t]]\) is not \(\exists \)-definable in \(E((t))\). Then \(F[[t]]\) is not \(\exists \)-definable in \(F((t))\).

In the next corollary, we apply the proposition and lemma to obtain negative results for a wide range of residue fields \(F\).

**Corollary**

\(F[[t]]\) is not \(\exists \)-definable in \(F((t))\) in each of the following cases:

1. \(F\) contains an algebraically closed field.
2. \(F\) is formally real and contains a real closed field.
3. \(F\) is formally \(p\)-adic and contains a \(p\)-adically closed field.
4. \(F\) is of characteristic zero and admits a non-trivial henselian valuation.
5. \(F\) is a purely transcendental extension of a proper subfield.

**Existential definitions: positive results**

The next theorem summarizes the examples for which we have - so far - found \(\exists \)-definitions.

**Theorem**

The valuation ring \(F[[t]]\) of \(F((t))\) is \(\exists \)-definable in each of the following cases:

1. \(F\) is finite.
2. \(F\) is PAC and does not contain an algebraically closed field.
3. \(F\) is PAC and does not contain a real closed field.
4. \(F\) is a number field.

**Proof idea.**

1 is from [1], 2 and 3 are from [3]. To prove 4 we use tools from [5]. Choose \(p\) so that \(F \subseteq Q_p\). Obtain an \(\exists \)-predicate \(R\) from [5] such that \(Z_1(p) \subseteq R(Q) \subseteq R(Q_p) \subseteq Z_p\). Show that \(R(F((t))) \subseteq F[[t]]\). Combining this with a few standard techniques, we can define \(F[[t]]\).

**Universal definitions**

When we turn to universal definitions the situation becomes surprisingly simple, at least when we allow parameters from the embedded residue field \(F\). First we give a definition from field arithmetic, due to Pop.

**Definition ([7])**

\(F\) is \(\bigwedge\) large if and only if every smooth curve defined over \(F\) either has no \(F\)-rational points or infinitely many \(F\)-rational points.

**Fact ([7])**

\(F\) is \(\bigwedge\) large if and only if \(F \preceq F((t))\) in the language of rings.

**Theorem (Characterisation of \(\bigwedge\)-definability)**

\(F((t))\) is \(\forall \bigwedge\)-definable in \(F((t))\) if and only if \(F\) is not large.

**Proof idea.**

\((\iff)\) If \(O = F[[t]]\) is \(\forall \bigwedge\)-definable then \(M = tF[[t]]\) is \(\exists \bigwedge\)-definable, and this shows that \(F \not\preceq F((t))\).

\(\implies\) If \(F\) is not large then there is a smooth curve \(C\) defined over \(F\) with infinitely many \(F\)-rational points. Using this set we can \(\exists \)-define a set \(X\) which contains \(M\) and is contained in finitely many residue classes of \(M\). With a few simple tricks this gives us an \(\exists \)-definition of \(M\) and thus a \(\exists \)-definition of \(O\), as required.

**Questions**

1. Can we find a characterisation of \(\forall \bigwedge\)-definability?
2. Can we find further examples of henselian valuation rings which are \(\exists \bigwedge\)-definable?

**References**

[1] Will Anscombe and Jochen Koenigsmann. An \(\exists \)-definition of \(F((t))\) in \(F[[t]]\). 